PERGAMON

International Journal of Heat and Mass Transfer 44 (2001) 4703–4710

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Transient laminar natural convection along rectangular ducts

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Received 17 October 2000; received in revised form 2 February 2001

Abstract

An analytical approach is proposed to investigate the transient behaviour of a Newtonian, single-phase fluid in natural laminar convection, in a rectangular open-ended duct. The continuity, momentum and energy equations, with the classical Boussinesq approximation, are solved using a twofold sine Fourier transform and the Laplace transform. The unsteady state is due to a step variation of the temperature in the four walls of the duct, which can assume four different, uniform arbitrary values. Considering hydrodynamically developed flow and uniform wall temperatures (UWTs), the velocity and temperature of the fluid are given as series containing two spatial co-ordinates and the time. Some plots show the transient evolution of the velocity and temperature distribution. Then the induced volumetric flow rate, the exchanged power, the mixing cup temperature, and the average Nusselt number are evaluated, as a function of time, emphasising the influence of the duct aspect ratio and the irrelevance of the channel height.

1. Introduction

Free laminar convection in inclined rectangular ducts open at both ends appears in many practical and industrial devices and its investigation constitutes a fundamental study of heat transfer process in thermal science. Examples of applications are found in the cooling of electronic circuit boards and electric transformers, architectural design for building insulation, solar heating and ventilating passive systems, geothermal systems, emergency cooling systems in inherently safe nuclear reactors [1]. The laminar motion is due to small fluid velocity and duct hydraulic diameter.

Being a fundamental topic of thermal science, transient natural convection has been extensively analysed in the past and even nowadays several papers are published by many authors, who improve and extend numerical and analytical solutions in different geometries, with different boundary and initial conditions.

Unsteady developing laminar free convection between vertical plates was numerically tackled by Kettleborough [2], who considered a step variation in the wall temperature. Joshi [3] solved numerically the problem of developing laminar flow along vertical parallel plates with uniform wall temperature (UWT) and uniform heat flux. Wang [4] carried out an analytical approach for fully developed transient flow in vertical plates with periodic heat input. Analytical solutions were presented by Al-Nimr and El-Shaarawi [5], using Green's function method; they estimated the transient volumetric flow rate, the mixing cup temperature and local Nusselt numbers, for fully developed flows. Nelson and Wood presented numerical [6] and analytical [7] solutions for natural convection heat and mass transfer between parallel plates. More recently, Lee [8] published a combined numerical and theoretical investigation for parallel plates, solving the system of balance equations with a finite difference approximation.

Cylindrical geometry has also been widely studied, vertical concentric annuli have been considered by Al-Nimr [9] and the transient equations have been solved with four different boundary conditions. Al-Shaarawi and Negm [10] have offered a finite difference solution for transient conjugate natural convection in vertical annuli with a step change in the temperature of the outer surface. Transient natural convection in rectangular

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0017-9310/01/$ - see front matter
PII: S0017-9310(01)00114-4

Nomenclature

a, b	longer and shorter sides of the rectangular cross-section (m)
c_p	fluid specific heat (J kg^{-1} K^{-1})
D	hydraulic diameter of the duct, $2ab/(a+b)$ (m)
F	dimensionless volumetric flow rate, defined in Eq. (25)
F'	volumetric flow rate (m^3 s^{-1})
F_∞	asymptotic dimensionless volumetric flow rate
F_r	dimensionless volumetric flow rate, F/F_∞
g	gravitational acceleration (m s^{-2})
Gr	Grashof number
h	convective heat transfer coefficient (W/m^2 K)
H	duct length (m)
k	fluid thermal conductivity (W/m K)
L	dimensionless duct length
Nu	average Nusselt number
p	pressure of the fluid in the duct (Pa)
p_0	fluid pressure at the channel entrance (Pa)
p_s	hydrostatic pressure in the ambient (Pa)
P	dimensionless pressure
Pr	Prandtl number
Q	dimensionless heat flow rate transferred between the walls and the fluid
Q'	heat flow rate transferred between the walls and the fluid (W)
Ra	Rayleigh number, $Pr\,Gr$
Re	Reynolds number, WD/v
t	time (s)
T_0	ambient and initial (fluid and wall) temperature (K)
T_w	average wall temperature (K)
$u(\cdot)$	axial fluid velocity (m s^{-1})
$U(\cdot)$	dimensionless fluid velocity
x, y, z	dimensionless rectangular Cartesian co-ordinates

Greek symbols

α	aspect ratio, $b/a \leqslant 1$
β	coefficient of thermal expansion (K^{-1})
v	kinematic viscosity (m^2 s^{-1})
ρ_0	fluid density at temperature T_0 (kg m^{-3})
θ	dimensionless temperature
τ	dimensionless time
ξ, η, ζ	Cartesian co-ordinates (m)

Subscripts

b	mixing cup (or bulk)
w	wall

channels constitutes a very arduous problem because of the geometrical complexity, which requires the introduction of one additional co-ordinate, with respect to the problem for parallel plates or concentric annuli. Laminar natural convection in open vertical rectangular ducts has recently been investigated by Lee [11], but only in steady state, using the axial vorticity function to solve numerically the energy and mass balance equations, with UWT or uniform heat flux boundary conditions.

The lack of analytical or numerical solutions for transient hydrodynamically developed laminar natural convection in rectangular ducts motivated this work. Hydrodynamically developed flow occurs in the channel if the Rayleigh number is low or when the height to diameter ratio *H/D* is sufficiently large (as in many practical applications). Laminar flow occurs when the temperature differences are small, so the surface normal velocity can be ignored.

Aim of this paper is the analytical solution, in closed form, for transient hydrodynamically developed laminar natural convection in vertical open channels of rectangular cross-section with four isothermal walls at four different temperatures.

The results constitute an original development in the field of thermal science and could provide a new tool for applications in several engineering problems.

2. Theoretical model

As usual in natural convection problems, a Newtonian single-phase fluid is considered and the Boussinesq approximation is assumed (neglecting density variation in the inertial terms of the balance equation and retaining it only in the buoyancy term of the motion equation). The fluid is in continuum laminar internal flow in an open rectangular channel; viscous dissipation, radiative heat transfer, axial conduction, and internal heat generation are absent. Fluid properties, except density in the buoyancy term, are constant. The Cartesian system of co-ordinates ξ, η, ζ has its origin in the left bottom of the rectangular cross-section (ξ and η in the cross-section, ζ perpendicular to the cross-section).

The initial condition states that the fluid is stagnant in the channel, with uniform initial temperature T_0, the walls have the same ambient temperature T_0. Suddenly, at time $t = 0$, the four walls, independently, undergo a step change in their temperature. If the walls are partially heated and cooled, the fluid flows upward near the hot walls, downward near the cold walls, as typical in natural convection. Even if no axial mass and heat flow rates occur, the fluid presents 2D velocity and temperature distributions.

Introducing the hypothesis of hydrodynamically developed flow, the axial velocity profile remains invariant with respect to the longitudinal co-ordinate ζ ($\partial u/\partial\zeta = 0$), and its distribution depends on the transverse co-ordinates ξ, η and time t. For the ambient fluid the hydrostatic law states that $\partial p_s/\partial\zeta = \pm\rho_0 g$, where the plus and minus signs are for downward (cooling) and upward (heating) flows, respectively. Based on the above assumptions, the equations of continuity, momentum and energy can be described by the two simultaneous dimensionless boundary-layer equations:

$$\frac{1}{Pr}\frac{\partial U}{\partial\tau} = \theta - \frac{1}{Gr}\frac{\partial P}{\partial z} + \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial\theta}{\partial\tau} + \frac{Ra}{16}U\frac{\partial\theta}{\partial z} = \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right). \tag{2}$$

The dimensionless independent variables (co-ordinates and time) are:

$$x = \frac{\xi}{a} \ (0 \leqslant x \leqslant 1), \quad y = \frac{\eta}{a} \ (0 \leqslant y \leqslant \alpha), \quad z = \frac{\zeta}{D}, \quad \tau = \frac{4kt}{\rho_0 c_p D^2}, \tag{3}$$

where the Grashof number is

$$Gr = \frac{g\beta D^3|T_w - T_0|}{\nu^2}. \tag{4}$$

The dimensionless dependent variables (pressure, velocity and temperature) are:

$$P = (p - p_s)\frac{D^2}{\rho_0\nu^2}, \quad U = u\frac{4D}{Gr\,\nu}, \quad \theta = \frac{T - T_0}{T_w - T_0}. \tag{5}$$

The initial and boundary conditions (no-slip conditions) for the fluid velocity are:

$$U(x,y,0) = U(0,y,\tau) = U(1,y,\tau) = U(x,0,\tau) = U(x,\alpha,\tau) = 0 \tag{6}$$

The boundary layer assumption leads to the conclusion that the pressure, in the vertical channel, depends (spatially) only on the axial co-ordinate z, hence the derivatives $\partial P/\partial x$ and $\partial P/\partial y$ are 0. The hypothesis of hydrodynamically developed flow implies $\partial U/\partial z = 0$. With this in mind, substituting θ from Eq. (1) into Eq. (2), the following expression is obtained:

$$\begin{aligned}\left(\frac{1+\alpha}{\alpha}\right)^2&\left[\frac{1}{Pr}\frac{\partial^2 U}{\partial\tau^2} + \frac{1}{Gr}\frac{\partial^2 P}{\partial z\partial\tau} - \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\partial^3 U}{\partial\tau\partial x^2} + \frac{\partial^3 U}{\partial\tau\partial y^2}\right) + \frac{UPr}{16}\frac{\partial^2 P}{\partial^2 z}\right] \\ &= \frac{1}{Pr}\frac{\partial^3 U}{\partial\tau\partial^2 x} - \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial^2 x\partial y^2}\right) \\ &\quad + \frac{1}{Pr}\frac{\partial^3 U}{\partial\tau\partial^2 y} - \left(\frac{\alpha}{1+\alpha}\right)^2\left(\frac{\partial^4 U}{\partial x^2\partial^2 y} + \frac{\partial^4 U}{\partial y^4}\right).\end{aligned} \tag{7}$$

The existence of a solution to Eq. (7) for $U(x,y,\tau)$ implies that the dimensionless pressure must satisfy the system of partial differential equations

$$\frac{\partial^2 P}{\partial z\partial\tau} = F_1(\tau) \quad \text{and} \quad \frac{\partial^2 P}{\partial z^2} = F_2(\tau), \tag{8}$$

where F_1 and F_2 are suitable time functions. Integrating the first of these equations with respect to time and deriving the result with respect to z we obtain

$$\frac{\partial P}{\partial z} = F_3(\tau) + G(z) \quad \text{and} \quad \frac{\partial^2 P}{\partial z^2} = G'(z), \tag{9}$$

where $F_3(\tau)$ is a general integral of F_1 and $G(z)$ is a suitable function depending only on z.

The second equations in (8) and (9) state that $G'(z) = F_2(\tau)$; this is possible only if $F_2(\tau)$ is a numerical constant f and P has a parabolic distribution along z. The dimensionless pressure is then

$$P = f\frac{z^2}{2} + F_4(\tau)z + F_5(\tau), \tag{10}$$

where the number f and the functions F_4 and F_5 depend on the boundary and initial conditions.

Eq. (10) is simply reduced considering that the channel is open, hence $P = 0$ at both inlet and exit sections ($z = 0$ and $z = L$). Consequently, $F_5(\tau) = 0$ and $F_4(\tau) = -fL/2$; the dimensionless pressure, under the above mentioned hypotheses, does not depend on time and it reads as

$$P = fz\frac{z - L}{2}. \tag{11}$$

Substituting Eq. (11) into Eq. (1) and deriving with respect to z (remembering the hydrodynamically developed flow) we obtain $\partial\theta/\partial z = f/Gr$, i.e. the temperature variation along the channel is constant at any point (x, y) of the rectangular cross-section and θ varies linearly with the axial distance z.

This implies that in a rectangular duct, in transient laminar natural convection and UWTs, if the flow is hydrodynamically developed, then it is thermally developed too, because $\partial(T - T_w)/(T_b - T_w)/\partial z = 0$. Hence the thermal entry length is never greater than the hydrodynamic development length, regardless of the value of the Prandtl number.

A further simplification can be attained if one considers that the temperatures of the four wetted walls are uniform (UWT boundary conditions). In order to satisfy this condition, the number f must be zero; consequently the dimensionless temperature profile depends only on the transverse co-ordinates x, y and time τ. These conclusions were already obtained by several authors for transient laminar flow in annuli or parallel plates, with UWT conditions [5,9].

The analysis is now simplified and the pressure term vanishes ($P = 0$) as well; the fluid pressure in the channel is equal to the hydrostatic pressure outside the duct. Eqs. (1) and (2) can now be separated as

$$\frac{1}{Pr}\frac{\partial U}{\partial \tau} = \theta + \left(\frac{\alpha}{1+\alpha}\right)^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \tag{12}$$

$$\frac{\partial \theta}{\partial \tau} = \left(\frac{\alpha}{1+\alpha}\right)^2 \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right), \tag{13}$$

where the energy equation contains only the unknown function θ, while the momentum equation links the velocity to the temperature.

Moreover, the initial and boundary conditions for the temperature profile (depending only on x, y and τ) are:

$$\theta(x,y,0) = 0, \quad \theta(0,y,\tau) = \theta_1, \quad \theta(1,y,\tau) = \theta_2, \\ \theta(x,0,\tau) = \theta_3, \quad \theta(x,\alpha,\tau) = \theta_4, \tag{14}$$

where θ_1 and θ_2 are the arbitrary uniform dimensionless temperatures of the short sides of the wetted perimeter, θ_3 and θ_4 are the uniform temperatures of the rectangular cross-section. Three of them are independent, being $\theta_w = 1$, hence $\alpha(\theta_1 + \theta_2) + \theta_3 + \theta_4 = 2(1+\alpha)$.

Obviously, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$ if the four walls have the same temperature.

3. Analytical solution

Eq. (13) is solved using the Fourier sine transform [12] with respect to the two spatial co-ordinates. By multiplying every term of the equation by $\sin(n\pi x)\sin(m\pi y/\alpha)$ and integrating in the whole domain (over $[0,1]$ along x and over $[0,\alpha]$ along y) the partial differential equation (13) is turned into a simple ordinary differential equation for the transformed temperature, which reads as (with its initial condition)

$$\frac{d\tilde{\theta}_{mn}(\tau)}{\partial \tau} = -B_{mn}\tilde{\theta}_{mn}(\tau) + A_{mn}, \quad \tilde{\theta}_{mn}(\tau = 0) = 0. \tag{15}$$

The twofold transformed temperature is

$$\tilde{\theta}_{mn}(\tau) = \int_0^1 \int_0^\beta \theta(x,y,\tau)\sin(n\pi x)\sin\left(\frac{m\pi y}{\alpha}\right), \tag{16}$$

and the numerical constants are

$$A_{mn} = \left(\frac{\alpha}{1+\alpha}\right)^2 \left\{\frac{\alpha n}{m}[\theta_1 - (-1)^n\theta_2][1-(-1)^m] + \frac{m}{\alpha n}[1-(-1)^n][\theta_3 - (-1)^m\theta_4]\right\}, \tag{17}$$

$$B_{mn} = \pi^2 \frac{\alpha^2 n^2 + m^2}{(1+\alpha)^2}. \tag{18}$$

The solution to Eq. (15) is easily obtained as

$$\tilde{\theta}_{mn}(\tau) = \frac{A_{mn}}{B_{mn}}(1 - e^{-B_{mn}\tau}). \tag{19}$$

Hence, the temperature field in the rectangular channel is

$$\theta(x,y,\tau) = \frac{4}{\alpha}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_{mn}}{B_{mn}}(1 - e^{-B_{mn}\tau}) \times \sin(n\pi x)\sin\left(\frac{m\pi y}{\alpha}\right). \tag{20}$$

This temperature distribution does not depend on the fluid properties (*Pr*, *Gr*). This expression for θ can be put in the momentum equation (12), which can be tackled resorting to both the Fourier sine transform (with respect to the spatial co-ordinates, using the same procedure described above) and the Laplace transform (with respect to time). After boring algebraic passages the velocity distribution is obtained as

$$U(x,y,\tau) = \frac{4}{\alpha}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{Pr}{Pr-1}\frac{C_{mn}}{B_{mn}} \times \left(\frac{Pr-1}{Pr} - e^{-B_{mn}\tau} + \frac{e^{-PrB_{mn}\tau}}{Pr}\right) \times \sin(n\pi x)\sin\left(\frac{m\pi y}{\alpha}\right). \tag{21}$$

If $Pr = 1$, the solution reads as

$$U(x,y,\tau) = \frac{4}{\alpha}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\left[\frac{1}{B_{mn}} - \left(\frac{1}{B_{mn}} + \tau\right)e^{-B_{mn}\tau}\right] \times \sin(n\pi x)\sin\left(\frac{m\pi y}{\alpha}\right), \tag{22}$$

where the constant C_{mn} is

$$C_{mn} = \frac{A_{mn}}{B_{mn}}. \tag{23}$$

While the temperature distribution depends only on the aspect ratio, the velocity profile depends on the Prandtl number too. At any point (x, y) in the channel, the fluid velocity u increases linearly with the Grashof number, as stated in the definition of dimensionless velocity U, Eq. (5).

It is interesting to point out that, in the steady state ($\tau \to \infty$) with uniform isothermal walls ($\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$), the velocity distribution is

$$U(x,y) = \frac{16(1+\alpha)^2}{\pi^4}\sum_{n=\text{odd}}^{\infty}\sum_{m=\text{odd}}^{\infty}\frac{1}{mn(\alpha^2 n^2 + m^2)} \times \sin(n\pi x)\sin\left(\frac{m\pi y}{\alpha}\right). \tag{24}$$

In steady-state, laminar, hydrodynamically developed flow, the velocity profile of Eq. (24) is formally quite similar to the forced convection profile [13], even if a direct comparison can not be proposed since the dimensionless velocity in forced convection is obtained using parameters that are zero in natural convection and vice versa.

4. Flow characteristics

The dimensionless induced volumetric flow rate defined as

$$F = \frac{8F'}{a\nu\, Gr} \tag{25}$$

can be estimated integrating the velocity distribution on the cross-sectional area

$$F(\tau) = \frac{1+\alpha}{\alpha} \int_0^1 \int_0^\alpha U \,\mathrm{d}x\,\mathrm{d}y. \tag{26}$$

When the developing natural convection flow reaches the steady-state condition ($\tau \to \infty$), the volumetric flow rate reaches its upper value; this asymptotic value does neither depend on the channel height nor the Prandtl number, but only on the aspect ratio of the channel. The flow rate F' is directly proportional to the Grashof number, as stated in Eq. (25). The exact expression of the asymptotic dimensionless flow rate, for UWT, is

$$F_\infty = \lim_{\tau\to\infty} F(\tau) = \frac{64(1+\alpha)^3}{\pi^6} \times \sum_{n=\mathrm{odd}}^{\infty} \sum_{m=\mathrm{odd}}^{\infty} \frac{1}{m^2 n^2(\alpha^2 n^2 + m^2)}. \tag{27}$$

The asymptotic volumetric flow rates are quoted in Table 1 as a function of the duct aspect ratio; in UWT this flow rate increases with the aspect ratio, reaching its maximum for the square channel.

The dimensionless mixing cup temperature is

$$\theta_\mathrm{b}(\tau) = \frac{1+\alpha}{\alpha F} \int_0^1 \int_0^\alpha U\theta \,\mathrm{d}x\,\mathrm{d}y, \tag{28}$$

while the dimensionless wall temperature is obviously $\theta_\mathrm{w} = 1$.

The dimensionless thermal power transferred between the fluid and the walls is

$$Q(\tau) = \frac{Q'}{\lambda H(T_\mathrm{w} - T_0)} = \left\{ \int_0^1 \left(-\left.\frac{\partial\theta}{\partial y}\right|_{y=0} + \left.\frac{\partial\theta}{\partial y}\right|_{y=\alpha} \right) \mathrm{d}x + \int_0^\alpha \left(-\left.\frac{\partial\theta}{\partial x}\right|_{x=0} + \left.\frac{\partial\theta}{\partial x}\right|_{x=1} \right) \mathrm{d}y \right\}. \tag{29}$$

At any time, the power exchanged does neither depend on Pr nor Gr. Finally the average Nusselt number (weighted on the whole heat exchange area) is determined as

$$Nu(\tau) = \frac{\alpha}{(1+\alpha)^2} \frac{Q}{1-\theta_\mathrm{b}}. \tag{30}$$

Since U and θ are the functions of x, y, τ and α, it follows that the flow rate, the mixing cup temperature, and the average Nusselt number are the functions of τ and α only (being related to definite integrals in x and y), regardless of the value of the axial co-ordinate. Hence the duct aspect ratio α becomes the fundamental parameter in order to establish the unsteady performance of the channel.

Table 1
The asymptotic value of the dimensionless volumetric flow rate as a function of the channel aspect ratio

α	F_∞
1	0.2812
0.9	0.2665
0.8	0.2504
0.7	0.2328
0.6	0.2136
0.5	0.1930
0.4	0.1711
1/3	0.1560
0.3	0.1485
0.25	0.1371
0.2	0.1259
0.1	0.1039

5. Results and discussion

The data related to the analytical solutions have been processed using double precision arithmetic in Fortran programming language, in a very short time using a PC class computer.

To show some examples of the results attained by the close-form solutions, Eqs. (20) and (21) are used to compute the fully developed transient temperature and velocity profiles in an isothermal square channel ($\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$) for an upward laminar flow (heating). In this case the constants A_{mn} are 0 for m or n even; for m and n odd they read as

$$A_{mn} = \left(\frac{2\alpha}{1+\alpha}\right)^2 \frac{n^2\alpha^2 + m^2}{mn\alpha}. \tag{31}$$

In Figs. 1 and 2 the temperature and velocity distributions are presented for different values of the dimensionless time; the numerical runs to compute velocity were performed for $Pr = 0.7$, typical for air flow (the temperature distribution does not depend on the Prandtl number), in the middle section $y = 0.5$.

As time increases, the step variation of the wall temperature propagates a perturbation in the core fluid, whose temperature increases until it reaches, for large values of τ, the steady-state value $\theta = 1$ at any given x and y co-ordinates (Fig. 1). The temperature distribution depends on τ and α only, its minimum always occurs at the centre of the cross-section, its maximum on the walls.

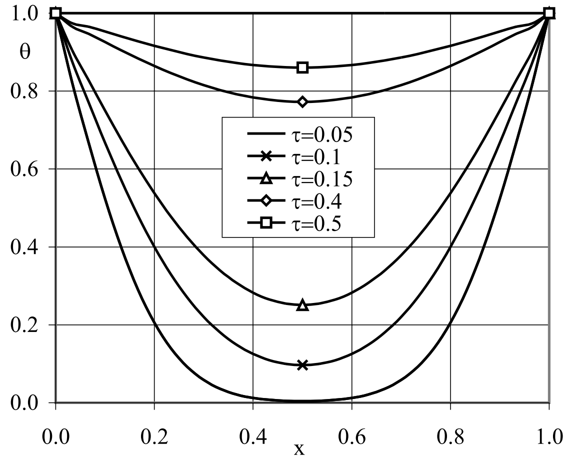

Fig. 1. Transient temperature distribution in a square channel, for $y = 0.5$.

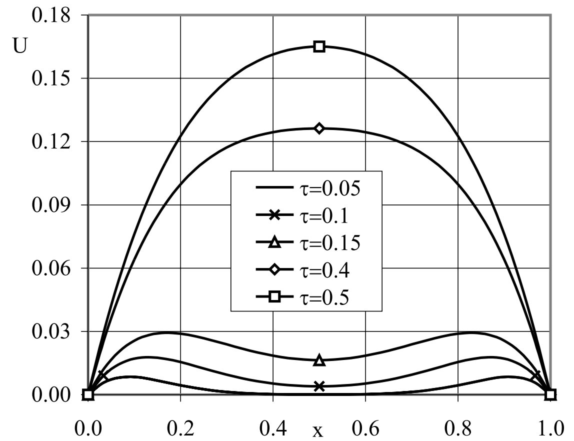

Fig. 2. Transient velocity distribution in a square channel, for $y = 0.5$ and $Pr = 0.7$.

The velocity spatial distribution in the square channel, shown in Fig. 2 for some values of τ, develops with time until it reaches the steady-state distribution. It is worthwhile to draw attention to a particular feature: at short times the velocity distribution has a local minimum in the centre of the cross-section and a pair of symmetrically placed maxima on either side of it. At long times the maxima disappear and the peak is reached at the centre of the rectangular cross-section. This is more evident for very low values of the Prandtl number (as usual in liquid metals), where the heat exchange is more effective. In fact the occurrence of a maximum near the hot wall is explained in terms of the effect of the step variation of the heated walls, which is felt more sensibly by the fluid near the walls. While the local maxima do not appear in Fig. 2, because the time intervals are not sufficiently small, they clearly appear in Fig. 3, for the same values of τ, where the Prandtl number is 0.025.

The relative induced volumetric flow rate F_r defined as F/F_∞, is quoted in Fig. 4, versus the dimensionless time τ, for different values of duct aspect ratio α. It is interesting to note that the relative volumetric flow rate increases if the aspect ratio diminishes. The asymptotic value of the volumetric flow rate is reached after a long time if the Prandtl number is small (this is the case of liquid metals, for instance). While the asymptotic value is constant, the time response is very sensitive on the Prandtl number. This is well explained by the physical meaning of Pr, related to the ratio between momentum and heat transport.

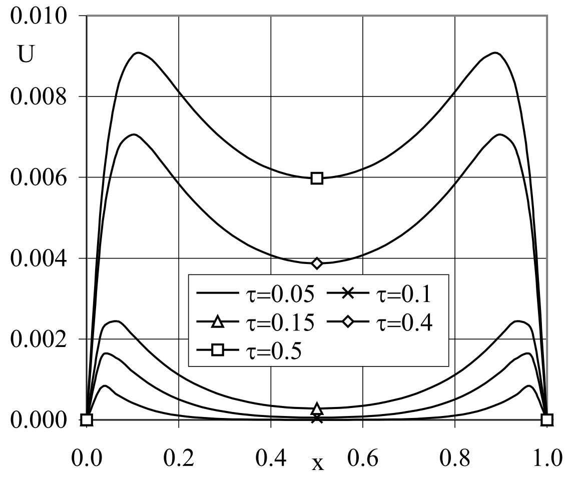

Fig. 3. Transient velocity distribution in a square channel, for $y = 0.5$ and $Pr = 0.025$.

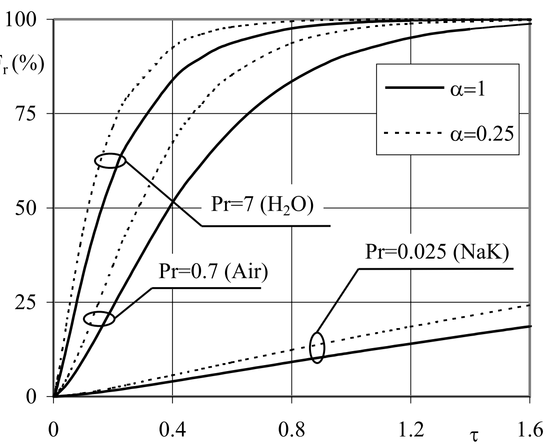

Fig. 4. Transient dimensionless induced flow rate in a rectangular channel, for $Pr = 7$, 0.7 and 0.025.

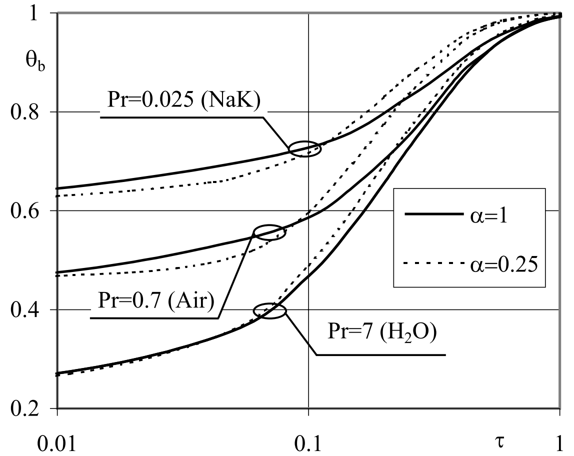

Fig. 5. Transient dimensionless mixing cup temperature in a rectangular channel, for $Pr = 7$, 0.7 and 0.025.

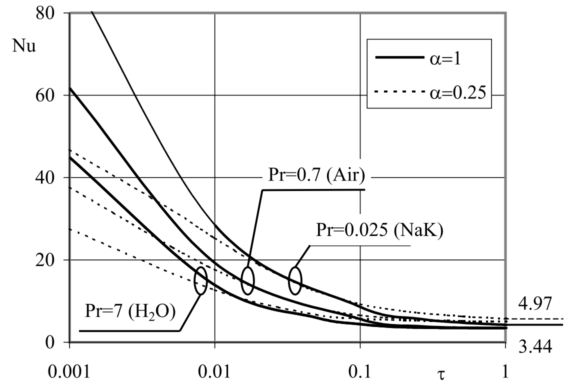

Fig. 6. Transient average Nusselt number in a rectangular channel, for $Pr = 7$, 0.7 and 0.025.

The mixing cup temperature θ_b, versus time for different aspect ratios and Prandtl numbers, is shown in Fig. 5. As expected, the fluids with low Prandtl numbers present a more rapid increase, even if the asymptotic value is always $\theta_b = 1$. In the square channel, greater values of temperature are experienced in the first part of the transient evolution, but the asymptotic value is reached with a time lag (with respect to rectangular ducts).

At last the average Nusselt number is sketched in Fig. 6 for different fluids and duct aspect ratio. It presents higher values for low Prandtl numbers and, for small times, for the square duct. Like in the analogous T boundary condition of forced convection [14], the Nusselt number in UWT natural convection reaches an asymptotic value depending only on the aspect ratio. This value corresponds to the indeterminate ratio between the heat flux and the temperature difference $T_w - T_b$; both 0.

6. Conclusions

The paper has analysed the behaviour of a Newtonian single-phase fluid in hydrodynamically developed transient natural laminar convection, in a rectangular vertical duct, with UWTs. Under these assumptions it is proved that:

- the flow is necessarily thermally developed, the thermal entry length cannot be greater than the hydrodynamic entrance length,
- no pressure drop occurs in the channel (the fluid viscous drag is offset by the buoyancy force),
- the dimensionless volumetric flow rate, the mixing cup temperature and the average Nusselt number are only time-dependent, they are not related to the channel height. Their time response depends on the Prandtl number, while their asymptotic value is constant (for θ_b) or depends only on the channel aspect ratio (for F and Nu).

The transient 2D velocity, and temperature have been analytically determined resorting to the Fourier and Laplace transforms. The transient temperature distribution depends on the aspect ratio, while the velocity distribution depends on the Prandtl number too, and increases linearly with the Grashof number.

Acknowledgements

Support from the Italian CNR and MURST is gratefully acknowledged.

References

[1] M. Caira, M. Cumo, A. Naviglio, The MARS nuclear reactor plant: an inherently safe, small/medium size multipurpose nuclear plant, Nucl. Eng. Des. 97 (1986) 145–160.
[2] C.F. Kettleborough, Transient laminar free convection between heated vertical plates including entrance effect, Int. J. Heat Mass Transfer 15 (1972) 883–896.
[3] H.M. Joshi, Transient effects in natural convection cooling of vertical parallel plates, Int. Commun. Heat Mass Transfer 15 (1988) 227–238.
[4] C.Y. Wang, Free convection between vertical plates with periodic heat input, J. Heat Transfer 110 (1988) 508–511.
[5] M.A. Al-Nimr, M.A. El-Shaarawi, Analytical solutions for transient laminar fully developed free convection in vertical channels, Heat Mass Transfer 30 (1995) 241–248.
[6] D.J. Nelson, B.D. Wood, Combined heat and mass transfer natural convection between vertical parallel plates, Int. J. Heat Mass Transfer 32 (1989) 1779–1787.
[7] D.J. Nelson, B.D. Wood, Fully developed combined heat and mass transfer natural convection between vertical parallel plates with asymmetric boundary conditions, Int. J. Heat Mass Transfer 32 (1989) 1789–1792.

[8] K.T. Lee, Natural convection heat and mass transfer in partially heated vertical parallel plates, Int. J. Heat Mass Transfer 42 (1999) 4417–4425.
[9] M.A. Al-Nimr, Analytical solution for transient laminar fully developed free convection in vertical concentric annuli, Int. J. Heat Mass Transfer 36 (1993) 2385–2395.
[10] M.A. El-Shaarawi, A.A.A. Negm, Transient combined natural convection–conduction in open-ended vertical concentric annuli, Heat Mass Transfer 35 (1999) 133–141.
[11] K.T. Lee, Laminar natural convection heat and mass transfer in vertical rectangular ducts, Int. J. Heat Mass Transfer 42 (1999) 4523–4534.
[12] B. Davies, Integral Transforms and their Applications, Springer, New York, 1985.
[13] M. Spiga, G.L. Morini, A symmetric solution for velocity profile in laminar flow through rectangular ducts, Int. Commun. Heat Mass Transfer 21 (4) (1994) 469–475.
[14] R.K. Shah, A.L. London, Laminar Flow Forced Convection in Ducts, Academic Press, New York, 1978.